

# Electric Time in Quantum Cosmology

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## Abstract

Effective quantum cosmology is formulated with a realistic global internal time given by the electric vector potential. New possibilities for the quantum behavior of space-time are found, and the high-density regime is shown to be very sensitive to the specific form of state realized.

## 1 Introduction

Even in simple isotropic models, the high-curvature regime of (loop) quantum cosmology remains poorly understood. At high curvature one expects strong quantum effects sensitive to what state the universe is in, but the precise form of suitable states is unknown. (In this paper, we will show a new explicit example for this sensitivity.) The popular use of Gaussians or semiclassical states is hard to justify in this regime, and if one starts with a semiclassical state at low curvature and evolves toward larger curvatures, the quantum state depends on the dynamics in all cosmic phases passed through. Quantum ambiguities then prevent precise-enough knowledge of the state dynamics.

As a second important issue, the problem of time [1, 2, 3] remains unsolved, affecting the right choice of dynamics. The problem is usually evaded (but not solved) by using specific choices of global internal times which tend to be unrealistic near the big bang, such as a free massless scalar or dust. As part of the problem of time, it is not known how to transform quantum wave functions or entire Hilbert spaces between different internal times, and therefore results found with one choice of global internal time do not necessarily

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hold for other choices. But if they depend on what time is used, they cannot be considered physical.

As elsewhere in physics, effective equations provide a better handle on reliable predictions. For a canonical setting such as canonical quantum cosmology, such equations do not refer to entire wave functions but rather to moments of a state [4, 5]. Only a small number of moments, chiefly fluctuations and the covariance, is needed in semiclassical regimes, and as one evolves toward stronger quantum regimes, one can self-consistently check when higher moments become relevant. Effective techniques for quantum constraints [6, 7, 8] also allow the use of realistic local internal times, and one can change between different times by mere gauge transformations [9, 10, 11]. In this article, we will not make use of these latter techniques because they require rather involved discussions of constrained systems. Instead, we develop the effective framework of quantum cosmology for a new choice of internal time which is still global but more realistic at high density than a free massless scalar or dust: radiation. In this article we demonstrate that a more realistic choice of internal time arises from electric fields<sup>1</sup>.

## 2 Radiation Hamiltonian and effective dynamics

The Hamiltonian constraint for spatially flat isotropic Friedmann–Lemaître–Robertson–Walker models with radiation is

$$H = -\frac{3}{8\pi G\gamma^2}c^2\sqrt{|p|} + \frac{E^2}{\sqrt{|p|}} = 0 \quad (1)$$

with canonical gravitational variables  $(c, p)$  with  $|p| = a^2$  and  $c = -\gamma\dot{a}\text{sgn}(p)$  (with the Barbero–Immirzi parameter  $\gamma$  relevant in loop quantizations [16, 17]) in terms of the scale factor  $a$ , the derivative being by proper time. We have Poisson brackets  $\{c, p\} = 8\pi\gamma G/3$  while the momentum  $A$  of  $E$ ,  $\{A, E\} = 1$ , does not appear in the Hamiltonian. The matter part is determined by the electric field  $E = |\vec{E}|$ ,<sup>2</sup> assumed sufficiently small so as not to cause significant anisotropy. By writing the constraint in the form of a Friedmann equation, dividing  $H$  by  $|p|^{3/2}$ , one can easily confirm that the matter term provides the correct behavior for radiation: the  $E$ -term amounts to an energy density  $\rho = E^2/p^2 = E^2/a^4$  with  $E$  constant because  $H$  does not depend on the momentum  $A$  conjugate to  $E$ . (Alternatively, the Hamiltonian can be derived using the standard electric-field energy density:  $q_{ab}E^aE^b/\sqrt{\det q}$  reduces to  $|p|E^2/|p|^{3/2}$  with an isotropic spatial metric  $q_{ab} = |p|\delta_{ab}$ .)

The momentum of  $E$  is the electromagnetic vector potential and would contribute a non-zero term to  $H$  in the presence of a magnetic field. However, a magnetic field requires deviations from homogeneity for the rotation of  $\vec{A}$  to be non-zero. In the symmetric

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<sup>1</sup>The use of electric fields in early universe cosmology has been implemented in inflationary theories [12, 13, 14, 15]

<sup>2</sup>The absolute value is taken with the flat Euclidean metric,  $|\vec{E}| = \delta_{ab}E^aE^b$ , to keep the spatial metric  $q_{ab}$  as a physical degree of freedom independent of  $E$ .

context used here, the restriction to pure electric fields is therefore meaningful. Since the electric field is canonically conjugate to the vector potential  $A$ , which does not appear in the constraint,  $E$  is constant and  $A$  can be used as a global internal time. We will call this choice *electric time*. To realize  $A$ -evolution, we follow standard techniques of deparameterization and solve the constraint equation for the momentum

$$p_A = E(c, p) = \pm \sqrt{\frac{3}{8\pi G \gamma^2}} |c| \sqrt{|p|}. \quad (2)$$

As a function on the gravitational phase space  $(c, p)$ ,  $E(c, p)$  provides Hamiltonian equations of motion for the classical  $c(A)$  and  $p(A)$ ,

$$\frac{dc}{dA} = \{c, E(c, p)\} \quad \text{and} \quad \frac{dp}{dA} = \{p, E(c, p)\},$$

as well as the basis for the quantum Hamiltonian of effective equations with respect to  $A$ . To transform equations or solutions to proper time  $\tau$ , we can multiply all  $d/dA$  by  $dA/d\tau = \{A, H\} = 2E/\sqrt{|p|}$ , using (1). We confirm the correct classical equations of motion

$$\frac{da}{d\tau} = \frac{\text{sgn}(p)}{2\sqrt{|p|}} \frac{dp}{d\tau} = \frac{E}{p} \frac{dp}{dA} = \frac{E}{p} \{p, E\} = \mp \sqrt{\frac{8\pi G}{3}} \frac{E \text{sgn}(cp)}{\sqrt{|p|}} = -\text{sgn}(p) \frac{c}{\gamma}$$

substituting  $c$  for  $E$  in the last step, and

$$\begin{aligned} \frac{1}{a} \frac{d^2 a}{d\tau^2} &= -\frac{\text{sgn}(p)}{\gamma a} \frac{dc}{d\tau} = -\frac{2E}{\gamma p} \frac{dc}{dA} = -\frac{2E}{\gamma p} \{c, E\} = \mp \frac{E}{\gamma p} \sqrt{\frac{8\pi G}{3}} \frac{|c| \text{sgn}(p)}{\sqrt{|p|}} \\ &= -\frac{8\pi G}{3} \frac{E^2}{a^4} = -\frac{4\pi G}{3} (\rho + 3P) \end{aligned}$$

where we have substituted  $E$  for  $c$  in the second line and used the electromagnetic expressions for energy density  $\rho$  and pressure  $P = \frac{1}{3}\rho$  to compare with the standard acceleration equation. (In what follows, we will set  $8\pi G/3 = 1$  and  $\gamma = 1$ , so that  $\{c, p\} = 1$ .)

## 2.1 Effective dynamics

The sign in (2) determines whether one considers solutions of positive or negative frequency with respect to time  $A$ . Without loss of generality, we will use the negative choice, such that  $E(c, p) = -|c|\sqrt{|p|}$ . Moreover, we can choose a definite sign of  $p$  (the orientation of space as measured by a triad) because we will consider only the approach to small  $p$ , not a possible transition from positive to negative  $p$ , or vice versa. For such a transition in dynamical terms to be described reliably, the Planck regime of quantum gravity would have to be much better understood than is possible at present. (To describe the transition non-singularly, we would have to refer to wave functions subject to a difference equation in loop quantum cosmology [18, 19, 20].) We will work with positive  $p > 0$ . Finally,

since  $E(c, p)$  is a conserved quantity, the sign of  $c\sqrt{p}$  never changes dynamically and we can drop the absolute value, the two sign options here merging with the explicit  $\pm$  in (2). Even for quantum states, the fact that the  $A$ -Hamiltonian  $E(c, p)$  and its quantization are conserved means that the absolute value can be dropped, provided that the expectation value  $\langle \widehat{c\sqrt{p}} \rangle$  is much larger than its quantum fluctuations. Truncating the whole state to a support of definite sign on the spectrum not just of  $|\widehat{c\sqrt{p}}|$  but also of  $\widehat{c\sqrt{p}}$  then ensures that no opposite-sign solutions mix. (These sign issues are the same as in harmonic cosmology obtained with a free massless scalar [21, 22]. For more details, see these papers or [23]. For the construction of corresponding Hilbert spaces in deparameterized quantum cosmology, see [24].)

Following the procedure of canonical effective equations [4, 5], the quantum Hamiltonian  $E_Q$  is a function on the quantum phase space with coordinates given by expectation values and moments

$$\Delta(c^a p^b) = \langle (\hat{c} - \langle \hat{c} \rangle)^a (\hat{p} - \langle \hat{p} \rangle)^b \rangle_{\text{symm}} \quad (3)$$

of a state (using totally symmetric ordering). These variables allow a Poisson structure by extending

$$\{\langle \hat{A} \rangle, \langle \hat{B} \rangle\} = \frac{\langle [\hat{A}, \hat{B}] \rangle}{i\hbar}, \quad (4)$$

defined for all operators  $\hat{A}$  and  $\hat{B}$ , to products of expectation values by the Leibniz rule. For fluctuations  $(\Delta c)^2 = \Delta(c^2)$ ,  $(\Delta p)^2 = \Delta(p^2)$  and the covariance  $C_{cp} = \Delta(cp)$ , we have

$$\{(\Delta c)^2, (\Delta p)^2\} = 4C_{cp} \quad , \quad \{(\Delta c)^2, C_{cp}\} = 2(\Delta c)^2 \quad , \quad \{(\Delta p)^2, C_{cp}\} = -2(\Delta p)^2. \quad (5)$$

Semiclassical states are defined generally by the hierarchy  $\Delta(c^a p^b) \sim O(\hbar^{(a+b)/2})$  of the moments. This class of states is much more general than that of Gaussian wave functions, which would determine all moments in terms of at most two parameters. At the level of effective equations, sufficient generality of the states considered can thus be guaranteed, without giving rise to prejudices about the form of wave functions. With a semiclassical (or other) hierarchy, the set of infinitely many moments can be truncated to finitely many ones by approximation, allowing practical methods to study the approach to strong quantum regimes. High orders of the moments, if required, make the equations unwieldy, but the derivation as well as solutions of equations of motion for moments to rather high orders can be done with efficient computational codes [25]. Quantum corrections by higher moments are analogs of higher time derivatives in effective actions [26], amounting in quantum cosmology to important higher-curvature corrections.

The quantum Hamiltonian is a power series in the moments of  $c$  and  $p$ , obtained by Taylor expanding the quantized  $\langle \hat{E} \rangle = \langle E(\langle \hat{c} \rangle + (\hat{c} - \langle \hat{c} \rangle), \langle \hat{p} \rangle + (\hat{p} - \langle \hat{p} \rangle)) \rangle$  in  $\hat{c} - \langle \hat{c} \rangle$  and  $\hat{p} - \langle \hat{p} \rangle$ :

$$E_Q := \langle \hat{E} \rangle = E(\langle \hat{c} \rangle, \langle \hat{p} \rangle) + \sum_{a,b} \frac{1}{a!b!} \frac{\partial^{a+b} E(\langle \hat{c} \rangle, \langle \hat{p} \rangle)}{\partial \langle \hat{c} \rangle^a \partial \langle \hat{p} \rangle^b} \Delta(c^a p^b). \quad (6)$$

Since we assume an operator for  $\hat{c}$  to exist, we will obtain the quantum Hamiltonian of a Wheeler–DeWitt quantization, as opposed to a loop quantization where only exponentials

$\widehat{\exp(i\delta c)}$  exist, but no  $\hat{c}$  [27, 23]. (A modification of the classical dynamics by holonomy corrections would be required in the latter case.) Choosing totally symmetric ordering for  $\widehat{c\sqrt{p}}$  and expanding to quadratic terms with second-order moments, we have

$$E_Q = -c\sqrt{p} - \frac{C_{cp}}{2\sqrt{p}} + \frac{1}{8} \frac{(\Delta p)^2}{p^{3/2}} c + \dots, \quad (7)$$

abbreviating  $c = \langle \hat{c} \rangle$  and  $p = \langle \hat{p} \rangle$  without risk of confusion. To this order, the Poisson structure provides effective equations

$$\frac{dp}{dA} = \sqrt{p} - \frac{1}{8} \frac{(\Delta p)^2}{p^{3/2}} \quad (8)$$

$$\frac{dc}{dA} = -\frac{c}{2\sqrt{p}} + \frac{C_{cp}}{4p^{3/2}} - \frac{3}{16} \frac{(\Delta p)^2}{p^{5/2}} c \quad (9)$$

$$\frac{d(\Delta p)^2}{dA} = \frac{(\Delta p)^2}{\sqrt{p}} \quad (10)$$

$$\frac{dC_{cp}}{dA} = \frac{1}{4} \frac{(\Delta p)^2}{p^{3/2}} c \quad (11)$$

$$\frac{d(\Delta c)^2}{dA} = -\frac{(\Delta c)^2}{\sqrt{p}} + \frac{1}{2} \frac{cC_{cp}}{p^{3/2}} \quad (12)$$

derived from Hamiltonian equations of motion  $df(c, p, \Delta(\cdot))/dA = \{f, E_Q(c, p, \Delta(\cdot))\}$ .

## 2.2 Solutions

The equations (8) and (10) for  $p$  and  $(\Delta p)^2$  are not coupled to the other variables and can be solved separately. To do so, we introduce a new evolution parameter  $x$  by  $dx = p^{-1/2}dA$ , so that (10) is completely decoupled:  $d(\Delta p)^2/dx = (\Delta p)^2$  is solved by

$$(\Delta p)^2(x) = (\Delta p)_0^2 e^x \quad (13)$$

with initial values at  $x = 0$ . Inserting this solution in (8) and rewriting it for  $p^2$ , we have the inhomogeneous differential equation  $dp^2/dx = 2p^2 - \frac{1}{4}(\Delta p)_0^2 e^x$  solved by

$$p(x) = p_0 e^x \sqrt{1 - \frac{1}{4} \frac{(\Delta p)_0^2}{p_0^2} (1 - e^{-x})}. \quad (14)$$

The function is real and positive for all semiclassical values, and in fact in the whole range of  $(\Delta p)_0 \leq 2p_0$ . If  $(\Delta p)_0 > 2p_0$ , in which case we have to be much more careful trusting our effective equations but may still analyze (14) to suggest possible effects to be corroborated further,  $p(x)$  remains real only for  $x < -\ln(1 - 4p_0^2/(\Delta p)_0^2)$ .

### 2.2.1 Potential effects at large fluctuations

In this brief section we collect properties of our equations and solutions when fluctuations become large, keeping in mind that we would have to go to higher orders in effective equations to justify the implications. Nevertheless, it is interesting to see what the equations may indicate.

The first derivative of  $p$  by  $x$ ,

$$\frac{dp}{dx} = e^{x/2} \frac{(p_0^2 - \frac{1}{4}(\Delta p)_0^2) e^x + \frac{1}{8}(\Delta p)_0^2}{\sqrt{(p_0^2 - \frac{1}{4}(\Delta p)_0^2) e^x + \frac{1}{4}(\Delta p)_0^2}},$$

becomes zero at

$$\bar{x} = \ln \left( \frac{\frac{1}{8}(\Delta p)_0^2}{-p_0^2 + \frac{1}{4}(\Delta p)_0^2} \right),$$

which is real and finite for  $(\Delta p)_0^2 > 4p_0^2$ . Turning points — bounces or recollapses — therefore require strong quantum effects and large relative fluctuations.

We can see the option for turning points directly from the equations of motion, especially (8). When fluctuations become large, the moment terms in the quantum Hamiltonian provide new possibilities for potential turning points of  $p(A)$ . We have  $dp/dA = 0$  for  $(\Delta p)^2 = 8p^2$ , requiring large relative volume fluctuations. To test whether this point can be a minimum, potentially corresponding to a bounce, we compute the second derivative of  $p$  by  $A$ ,

$$\frac{d^2p}{dA^2} = \frac{1}{2} - \frac{3}{128} \frac{(\Delta p)^4}{p^4} \quad (15)$$

using the equations of motion. (All terms linear in  $(\Delta p)^2$  cancel.) For  $(\Delta p)^2 = 8p^2$ ,  $d^2p/dA^2 = -1/4$ , indicating a maximum of  $p$  and therefore a recollapse. However, if  $(\Delta p)^2$  in (15) is significant, higher moments may easily contribute and change the behavior. The precise form of the turning point found here is therefore an explicit example for an effect that is highly sensitive to the precise form of quantum state. For other examples in terms of wave functions, see the solutions given in [28, 29].

Quantum fluctuations could therefore trigger a recollapse, reminiscent of the effect pointed out in [30, 31]. At the turning point, using our solution (14), we have

$$p(\bar{x}) = \frac{\Delta p_0}{4\sqrt{1 - 4p_0^2/(\Delta p)_0^2}} > \frac{1}{2}p_0.$$

The recollapse value  $p(\bar{x})$  may be large if  $(\Delta p)_0^2$  is close to  $4p_0^2$ , and it is equal to  $p_0$  for  $(\Delta p)_0^2 = 8p_0^2$ . (If  $(\Delta p)_0^2 = 8p_0^2$ ,  $x < \ln 2$  in order for  $p(x)$  to be real.)

We can now transform back to  $A$  by integrating

$$dA = \sqrt{p}dx = \sqrt{p_0}e^{x/2} \sqrt{1 - \frac{1}{4} \frac{(\Delta p)_0^2}{p_0^2} (1 - e^{-x})} dx.$$

The result can be expressed in terms of hypergeometric functions. For instance, using for illustrative purposes the value  $(\Delta p)_0^2 = 8p_0^2$ , we find

$$A(x) = A_0 - 2\sqrt{p_0} e^{\frac{x}{4}} \sqrt[4]{2e^x - 1} \left[ 2 - e^x + \sqrt[4]{2}(2 - e^x)^{\frac{3}{4}} \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})\Gamma(\frac{1}{2})} \int_0^x \frac{dt}{t^{\frac{1}{4}} (1-t)^{\frac{1}{2}} (1-te^x)^{\frac{1}{4}}} \right], \quad (16)$$

in which  $\Gamma$  represents the Euler function.

Unfortunately, this function and especially its inversion for  $x(A)$ , to be inserted in  $p(x)$ , are complicated. We can proceed further with additional approximations. First, as one example to explore the strong quantum regime, we may assume that the initial value  $(\Delta p)_0^2$  is close to  $8p_0^2$ . Thus  $\bar{x} \rightarrow 0$ , and if we are interested in what happens close to  $x \sim 0$ , we can expand

$$\frac{dA}{dx} \sim \sqrt{p_0} \left( 1 - \frac{1}{4}x^2 \right) + O(x^3).$$

Integrating from 0 to  $x$  with  $A_0 = 0$ , we find

$$A(x) \sim \sqrt{p_0}x \left( 1 - \frac{1}{12}x^2 \right) + O(x^4), \quad (17)$$

inverted by

$$x = \frac{A}{\sqrt{p_0}} \left( 1 + \frac{1}{12} \left( \frac{A}{\sqrt{p_0}} \right)^2 \right) + O(A^4).$$

We can then go back to (14), expand it as  $p(x) = p_0(1 - \frac{1}{2}x^2) + O(x^3)$ , and find

$$p(A) = p_0 \left( 1 - \frac{1}{2} \left( \frac{A}{\sqrt{p_0}} \right)^2 \right) + O(A^3).$$

### 2.2.2 Small fluctuations

For small relative initial fluctuations  $(\Delta p)_0^2/p_0^2 \ll 1$ , for which our effective equations are reliable, and sufficiently small  $x$ , we can expand the square root and integrate  $dA \sim \sqrt{p_0}e^{x/2} \left( 1 - \frac{1}{16}((\Delta p)_0^2/p_0^2)(1 - e^{-x}) \right) dx$  to

$$\begin{aligned} A(x) &\sim 2\sqrt{p_0} \left( \left( 1 - \frac{1}{16} \frac{(\Delta p)_0^2}{p_0^2} \right) e^{x/2} - \frac{1}{16} \frac{(\Delta p)_0^2}{p_0^2} e^{-x/2} \right) + A_0 \\ &= 2\sqrt{p_0} \sqrt{1 - b^2} \sinh(x/2 + \text{arcosh}(1/\sqrt{1 - b^2})) + A_0 \end{aligned}$$

with  $b = 1 - \frac{1}{8}(\Delta p)_0^2/p_0^2$ . To express  $x$  in terms of  $A$ , we write

$$\begin{aligned} x &= 2\text{arsinh} \left( \frac{1}{2} \left( 1 - \left( 1 - \frac{1}{8}(\Delta p)_0^2/p_0^2 \right)^2 \right)^{-1/2} (A - A_0)/\sqrt{p_0} \right) \\ &\quad - \text{arcosh} \left( \left( 1 - \left( 1 - \frac{1}{8}(\Delta p)_0^2/p_0^2 \right)^2 \right)^{-1/2} \right). \end{aligned}$$

We emphasize that we had to assume relative fluctuations to be small only at one time, which could be in a semiclassical regime. Our solutions are then valid even in stronger quantum regimes (until, of course, higher moments grow large).

Finally, we can integrate all equations (8)–(12) perturbatively if we assume moments to be small throughout the whole evolution. At zeroth order, we first solve the classical equations, ignoring all moments. We obtain

$$p_{\text{classical}}(A) = (\sqrt{p_0} + A/2)^2 \quad \text{and} \quad c_{\text{classical}}(A) = \frac{c_0\sqrt{p_0}}{\sqrt{p_0} + A/2} \propto \frac{1}{\sqrt{p_{\text{classical}}(A)}} \quad (18)$$

with initial values  $p_0$  and  $c_0$  when  $A = 0$ . These solutions can then be assumed in the equations of motion for moments to find approximate solutions for the latter. We obtain

$$(\Delta p)^2(A) \propto p(A) \quad , \quad C_{cp}(A) \propto -c(A) + \text{const} \quad \text{and} \quad (\Delta c)^2(A) \propto c^4 + \text{const}'c^3 + \text{const}''c^2. \quad (19)$$

(The first of these equations is, of course, consistent with our full solutions (13) and (14) for small  $(\Delta p)_0^2/p_0^2$ .) In particular, relative fluctuations  $(\Delta p)^2/p^2 \propto p^{-1}$ ,  $C_{cp}/(cp) \propto p^{-1}$  and  $(\Delta c)^2/c^2 \propto \text{const}''$  remain small at small  $c$ . Moreover, the uncertainty product  $(\Delta c)^2(\Delta p)^2$  is bounded from below by  $\text{const}''$ , and the uncertainty relation will never be violated if we choose appropriate values for the constants.

These solutions indicate that effective equations get better and better when one evolves toward larger  $p$ , but give rise to strong quantum effects with growing relative fluctuations at small  $p$ .

### 3 Effective equations in terms of proper time

The transformation from internal times to proper time after quantization may not be obvious because it requires a careful look at quantum corrections of different but related expressions — the internal-time Hamiltonian and the Hamiltonian constraint. Nevertheless, once an internal time has been chosen, there is a unique procedure to transform equations or solutions back to proper time. We will first go through the general procedure to highlight ambiguities and difficulties of deparameterized quantizations, as part of the problem of time.

To obtain equations of motion in proper time in our classical deparameterized model, we transformed  $d/dA$  to  $d/d\tau = 2E/\sqrt{p}d/dA$  with constant  $E$ , computing  $dA/d\tau = \{A, H[N]\}$  with a Hamiltonian constraint  $H$  of lapse function  $N = 1$ . In the deparameterized quantum model, we quantize  $E(c, p)$  after having solved the Hamiltonian constraint equation for  $E$ ; we do not quantize  $H$  itself. After deparameterized quantization, we can go back to a Hamiltonian constraint

$$H_Q = \frac{E^2}{\sqrt{p}} - \frac{E_Q(c, p, \Delta(\cdot))^2}{\sqrt{p}} = \frac{E^2}{\sqrt{p}} - \frac{(c\sqrt{p} + \frac{1}{2}C_{cp}p^{-1/2} - \frac{1}{8}(\Delta p)^2cp^{-3/2} + \dots)^2}{\sqrt{p}} \quad (20)$$



with quantum corrections in  $E_Q(c, p, \Delta(\cdot))$ . By construction, this corrected Hamiltonian constraint gives rise to the deparameterized model we started with if the momentum  $A$  of  $E$  is chosen as time. We therefore transform  $A$ -derivatives to proper-time derivatives by using  $dA/d\tau = \{A, H_Q\} = 2E/\sqrt{p} \approx 2E_Q(c, p, \Delta(\cdot))/\sqrt{p}$ , the latter equation holding on shell when  $H_Q = 0$ .

With this procedure, the relationship between internal-time intervals  $dA$  and proper-time intervals  $d\tau$  in terms of  $E$  does not differ from the classical one, except that a new function  $E_Q$  is used. The identification of  $E$  with  $E_Q$ , without further quantum corrections, comes about because  $A$  has been chosen as internal time. As a time variable, it is not quantized and retains its classical form in quantum evolution equations. By construction, the momentum  $p_A$  of  $A$  at the quantum level is then  $E_Q$ , which is used in the electromagnetic Hamiltonian to compute the relation between  $A$  and proper time.

If we had quantized the Hamiltonian constraint and then solved it or deparameterized it *after* quantization, additional quantum corrections with moment terms from  $E^2/\sqrt{p}$  would have resulted. These corrections are not included in a deparameterized quantization or in a quantization of the corresponding reduced phase space. A single deparameterized quantization is consistent as long as one does not ask how it may be related to other possible deparameterizations with other choices of internal time. However, if one allows for different internal times, the resulting quantum theories are not likely to be equivalent: In each case, one first solves the classical Hamiltonian constraint equation  $H = 0$  for a different variable,  $E$  in the choice made here or some other degree of freedom in a different model. The resulting internal-time Hamiltonians then take into account quantum corrections which differ from each other and from the corrections that a quantization of the original Hamiltonian constraint would imply. Deparameterized quantizations not only ignore some quantum corrections in the Hamiltonian constraint, the terms ignored even differ depending on which internal time one chooses. Predictions of different models are then unlikely to be equivalent, and a single deparameterized model cannot be considered physical unless one can show how its results are related to those of different time choices. Our considerations in this paper, as well as most results in quantum cosmology that rely on quantization after deparameterization, therefore cannot be considered complete. Our aim here is not to derive complete effects but rather to give examples for the different features that various choices of internal time can give rise to.

After these cautionary remarks, we now continue with our discussion of electric time.

Multiplying our previous equations with  $2E_Q/\sqrt{p}$ , we obtain

$$\frac{dp}{d\tau} = E_Q \left( 2 - \frac{1}{4} \frac{(\Delta p)^2}{p^2} \right) = -2c\sqrt{p} - \frac{C_{cp}}{\sqrt{p}} + \frac{1}{2}(\Delta p)^2 \frac{c}{p^{3/2}} \quad (21)$$

$$\frac{dc}{d\tau} = E_Q \left( -\frac{c}{p} + \frac{C_{cp}}{2p^2} - \frac{3}{8} \frac{(\Delta p)^2}{p^3} c \right) = \frac{c^2}{\sqrt{p}} + \frac{1}{4}(\Delta p)^2 \frac{c^2}{p^{5/2}} \quad (22)$$

$$\frac{d(\Delta p)^2}{d\tau} = 2E_Q \frac{(\Delta p)^2}{p} = -2(\Delta p)^2 \frac{c}{\sqrt{p}} \quad (23)$$

$$\frac{dC_{cp}}{d\tau} = \frac{1}{2}E_Q \frac{(\Delta p)^2}{p^2} c = -\frac{1}{2}(\Delta p)^2 \frac{c^2}{p^{3/2}} \quad (24)$$

$$\frac{d(\Delta c)^2}{d\tau} = E_Q \left( -2 \frac{(\Delta c)^2}{p} + \frac{cC_{cp}}{p^2} \right) = 2(\Delta c)^2 \frac{c}{\sqrt{p}} - C_{cp} \frac{c^2}{p^{3/2}} \quad (25)$$

where, consistent with our approximation, we have ignored quadratic terms in the small second-order moments because they would compete with higher-order moments which are ignored here. As one can check explicitly, the moments satisfy

$$\frac{d}{d\tau} ((\Delta p)^2 (\Delta c)^2 - C_{cp}^2) = 0 \quad (26)$$

so that the uncertainty product is preserved. Any departure of a state from saturating the uncertainty relation

$$(\Delta p)^2 (\Delta c)^2 - C_{cp}^2 \geq \frac{\hbar^2}{4} \quad (27)$$

remains constant. If an initial state saturates the uncertainty relation, it will keep saturating it in this regime, and we obtain a dynamical coherent state.

From the first of these equations it follows that the classical relation  $c = -\frac{1}{2}\dot{p}/\sqrt{p} = -\dot{a}$ , which descends from the isotropic reduction of the Ashtekar–Barbero connection  $A_a^i = \Gamma_a^i - \gamma K_a^i$  ( $\gamma = 1$ ) for a spatially flat cosmological model, receives quantum corrections: writing  $p$  in terms of the scale factor, we have

$$c = -\frac{\dot{a} + \frac{1}{2}C_{cp}/p}{1 - \frac{1}{4}(\Delta p)^2/p^2} = -\dot{a} \left( 1 + \frac{1}{4}(\Delta p)^2/p^2 \right) - \frac{1}{2}C_{cp}/a^2 + \dots \quad (28)$$

(We cannot easily transform the moments to those of  $a$  and  $\dot{a}$  because these variables are non-linearly related to  $p$  and  $c$ .) With this relation, we can use the equation for  $dc/d\tau$  to compute the acceleration equation

$$\frac{\ddot{a}}{a} = - \left( 1 + \frac{1}{4} \frac{(\Delta p)^2}{p^2} \right) \left( \frac{\dot{a}}{a} \right)^2, \quad (29)$$

ignoring products of moments. For small relative fluctuations, the equation is consistent with the classical one for radiation. Only the  $p$ -fluctuation enters, while all covariance terms cancel. Even if fluctuations could be large, they would not cause acceleration.

It is interesting to analyze these equations near the time when  $\dot{a} = 0 = \dot{p}$  (which, as we recall, requires large fluctuations), where we have small  $c = -\frac{1}{2}C_{cp}/a^2$  and can ignore  $c^2$ -terms in our equations. In addition to  $a$ ,  $c$  is then nearly constant and we can integrate the moment equations

$$\frac{d(\Delta p)^2}{d\tau} = -2(\Delta p)^2 \frac{c}{a} \quad , \quad \frac{dC_{cp}}{d\tau} = 0 \quad , \quad \frac{d(\Delta c)^2}{d\tau} = 2(\Delta c)^2 \frac{c}{a} \quad (30)$$

to obtain

$$(\Delta p)^2(\tau) \sim (\Delta p)_1^2 \exp(-2(\tau - \tau_1)c/a) \quad , \quad (\Delta c)^2(\tau) \sim (\Delta c)_1^2 \exp(2(\tau - \tau_1)c/a) \quad (31)$$

where  $\tau_1$  is the time when  $\dot{a} = 0$ .

## 4 Numerical solutions

We return to the Hamiltonian system in (7), to seek self-consistent solutions. Here we only illustrate differences between classical and quantum solutions and test the validity of our approximations, which turn out to be very good. A detailed analysis especially near potential turning points could, as motivated by [32], reveal a wave function of the Universe that entails parity violation and non-trivial chiral effects. However, not just the numerics but also higher orders of our effective equations would then have to be developed.

We attempt to solve the system (7) with certain initial conditions for  $c(A)|_{A=A_0}$  and the other variables. We may assume the normalization of  $p(A)$  initially, such that  $p(A)|_{A=A_0} = 1$ . We then fix  $C_{cp}|_{A=A_0} = C_{cp}^0$ ,  $(\Delta p)_{A=A_0}^2 = (\Delta p)_0^2$ , and  $(\Delta c)_{A=A_0}^2 := (\Delta c)_0^2$ , respecting the uncertainty relation (27). Our effective equations are valid for some time provided that  $(\Delta p)_0^2 \ll p_0^2$ , and so on, but we will choose larger values just to illustrate the implications of quantum corrections. Even for  $(\Delta p)_0^2 \approx p_0^2$ , it turns out that we remain close to the classical solutions for rather long times.

We therefore solve the system (7) parametrizing solutions in terms of two positive real numbers, namely  $(\Delta p)_0^2, (\Delta c)_0^2 \in \mathbb{R}^+$ , and a real number  $C_{cp}^0 \in \mathbb{R}$ , which together with  $p(A)|_{A=A_0}$  and  $c(A)|_{A=A_0}$  represent our initial conditions for the system. The moments must obey the uncertainty relation (27). With these initial values, we solve (8)–(12).

Figure 1 shows good agreement with the classical solutions even for rather large fluctuations, with  $\Delta p_0 = p_0$  and  $\Delta c_0 = c_0$  ( $C_{cp} = 0$ ). Numerical solutions assuming  $c_0 = 0$  together with  $(\Delta p)_0^2 = (\Delta c)_0^2 = 1$  and  $C_{cp}^0 = 3/4$  are shown in Fig. 2. The plot for  $p(A)$  is increasing in  $A$ , with a certain rate amounting to the expansion of the universe. The plot for  $c(A)$  seems to indicate acceleration in the increasing branch, but this is realized only in terms of internal time  $A$ , not in terms of proper time (see (29)).

In Fig. 3, we see numerical evolutions of the second-order moments. The  $c$ -variance decreases in  $A$  and reaches quite soon a vanishing value. On the other hand, the  $p$ -variance increases with the same rate of  $p(A)$ . The covariance for the position and momentum operators is almost stable in the range studied, while the two fluctuations evolve according to (31). The good agreement with our analytical solutions and approximations is further illustrated in Figs. 4 and 5.

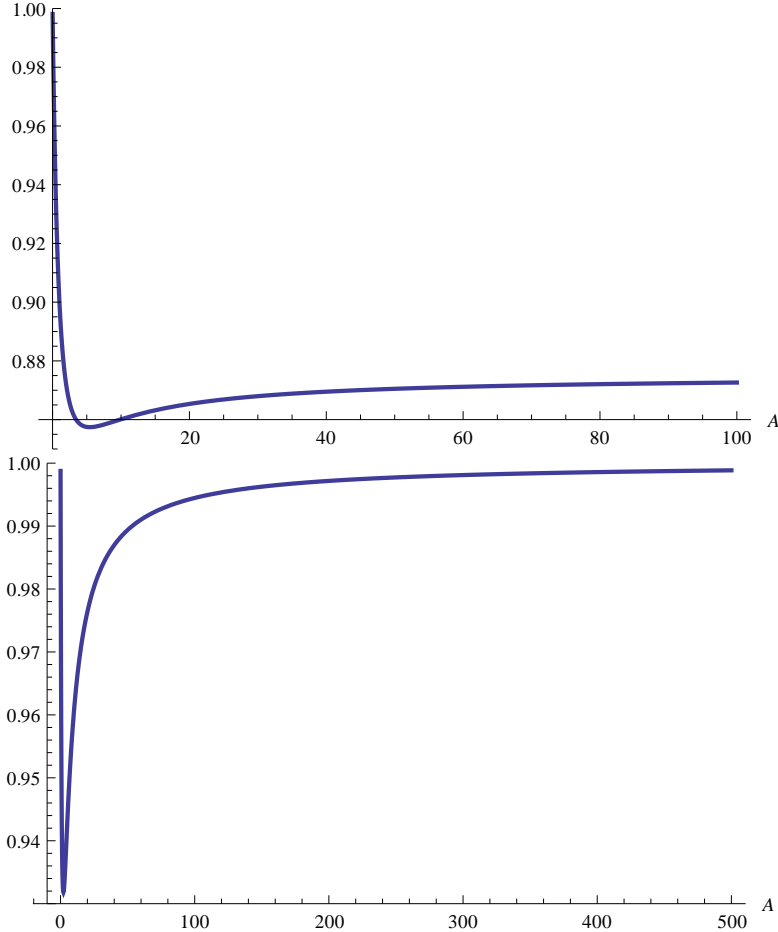


Figure 1: Solutions of effective equations for expectation values, plotted as their ratios to the classical solutions  $c_{\text{classical}}(A)$  (top) and  $p_{\text{classical}}(A)$  (bottom). Initial fluctuations have been set to rather large values —  $\Delta p_0 = \Delta c_0 = p_0 = c_0 = 1$  with  $C_{cp} = 0$  — to show the implications of quantum corrections more clearly. Nevertheless, the ratios stay close to one.

## 5 Conclusions

We have laid out the basic description of deparameterized quantum cosmology with time provided by the electric field. The only matter source required to formulate time evolution is radiation, expected to be significant in any early-universe model. No artificial matter sources such as dust or free massless scalars are required.

Our discussion in this paper does not include modifications suggested by loop quantum cosmology, such as holonomy and inverse-triad corrections. The former would prevent us from using  $c$  as an operator and in moments, making the Hamiltonian more non-linear by replacing  $c$  with functions such as  $\sin(\delta c)/\delta$ . Such modifications certainly alter the high-curvature behavior and are expected to compete with fluctuation terms. The magnitude of holonomy corrections depends on the parameter  $\delta$  used in this modification (a quantization

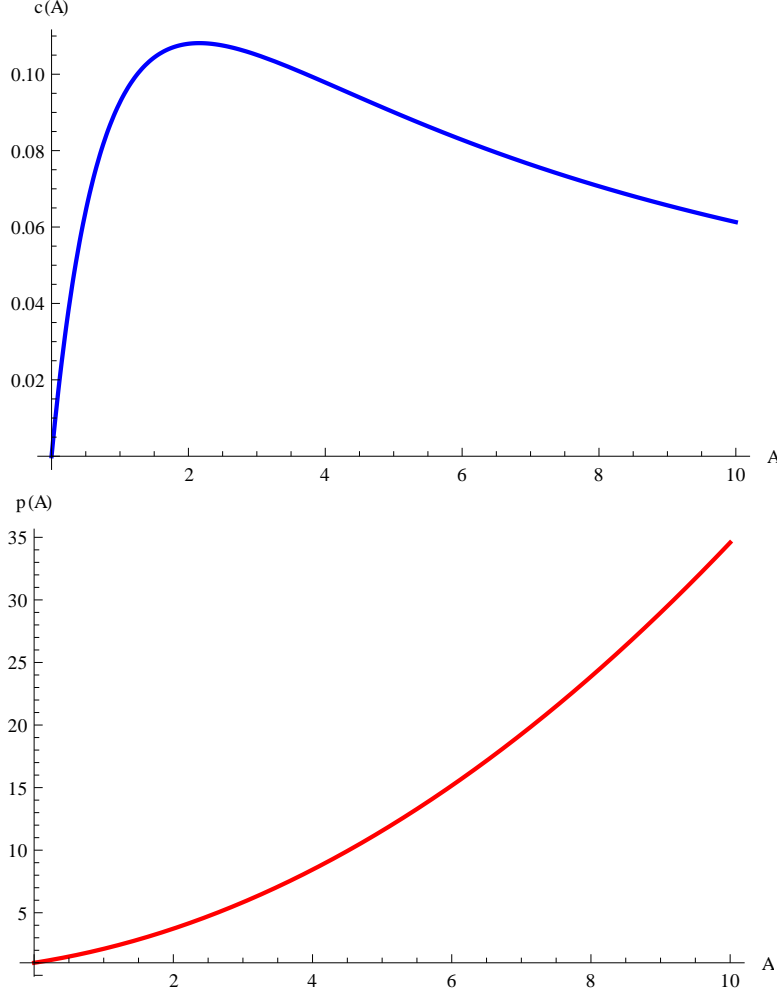


Figure 2: The dynamics in  $A$  of the expectation values on the quantum state of the universe of  $c(A)$  and  $p(A)$ , computed by solving the second-order equations (8)–(12) numerically. Here,  $c_0 = 0$  initially, and  $C_{cp} = 3/4$ , all other initial values as in Fig. 1.

ambiguity). However, if  $\delta$  is related to the Planck length, for instance by  $\delta \sim \ell_P/\sqrt{p}$  as often assumed, holonomy corrections are of the size  $\ell_P^2/\ell_H^2$  with the Hubble scale  $\ell_H^2$ , the same size expected for higher-curvature corrections. Quantum back-reaction by moments, on the other hand, contributes higher-derivative terms, also related to higher-curvature corrections. If a loop quantization is used, it is therefore impossible to study either holonomy corrections or quantum back-reaction by fluctuations in isolation. In this paper, to provide a manageable first discussion of quantum cosmology with electric time, we have therefore decided to ignore loop effects so that our solutions refer to quantum back-reaction in Wheeler–DeWitt models. (The alternative, ignoring quantum back-reaction but keeping holonomy corrections, is not consistent unless one restricts oneself to models in which quantum back-reaction is absent or weak. These are only the models of harmonic cosmology [21, 33] or kinetic-dominated ones [34, 35, 36].) Moreover, recent results in off-shell

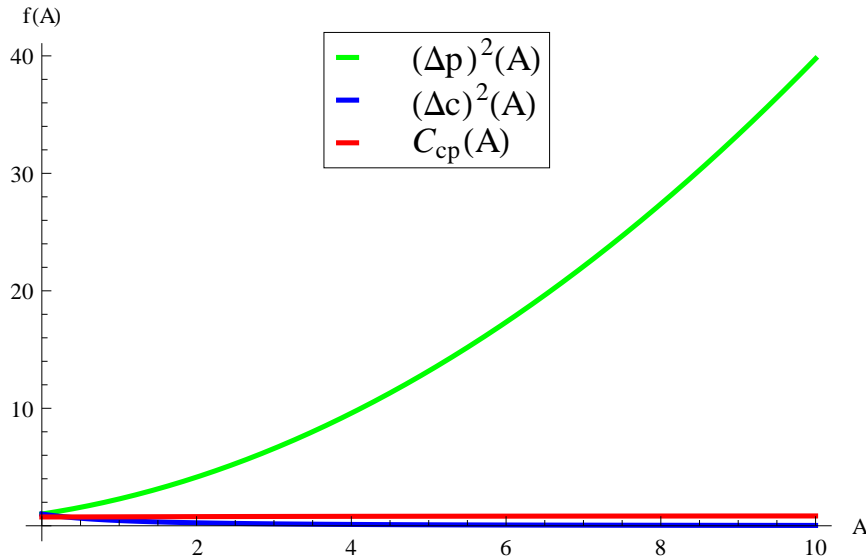


Figure 3: The variance of  $p$  (in green), of  $c$  (in blue), and the covariance between the two operators (in red) for a quantum state of the universe that minimizes the uncertainty relation (27). Good agreement for instance with (31) is realized. (Initial values as in Fig. 2.)

loop quantum gravity [37, 38, 39, 40, 41, 42, 43] have shown that holonomy corrections in loop quantum cosmology must be treated with care because they trigger signature change at high density [44]. Their evolution equations must therefore stop before interesting high-density effects can be realized.

Our discussion of the quantum dynamics was done mainly at the effective level of Wheeler–DeWitt quantum cosmology, allowing us to consider large classes of states without tying us down to specific wave functions (such as Gaussians). Considering quantum corrections by fluctuations, we have found several new possibilities for early-universe dynamics, showing also the high sensitivity of potential turning points to specific forms of states. The small-volume behavior in electric time is rather different from that with a free massless scalar as time. In the latter case, the singularity is approached exponentially with respect to the scalar (or by a power law in proper time) while electric time, which does not give rise to a harmonic model, leads to a more complicated behavior. Although we have pointed out the possibility of inequivalent quantum corrections obtained in different deparameterizations (Section 3), it remains to be seen whether these differences are solely due to the different matter sources or due to effects of choices of internal times. To decide this question, effective-constraint methods [6, 7] would be suitable but are rather complicated to perform. We therefore end this paper by concluding that much remains to be done before the high-curvature regime of quantum cosmology can be controlled, including a better understanding of the role of internal times.

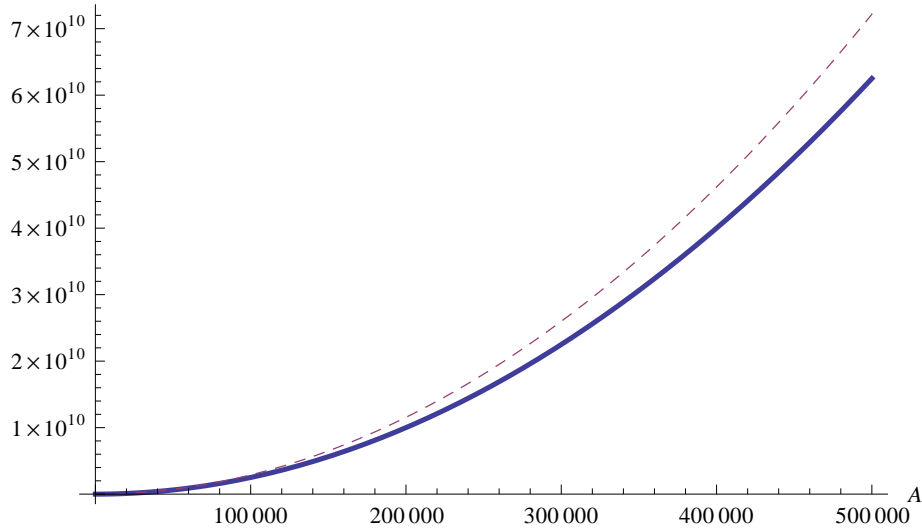


Figure 4: The squared  $p$ -variance  $(\Delta p)^2(A)$  (dashed) compared with the expectation value  $p(A)$  (solid). Both functions increase in nearly the same way even for large initial fluctuations as in Fig. 1, confirming our analytical solutions. We have  $(\Delta p)^2 > p$ , but relative fluctuations  $(\Delta p)^2/p^2$  become very small at large  $A$ .

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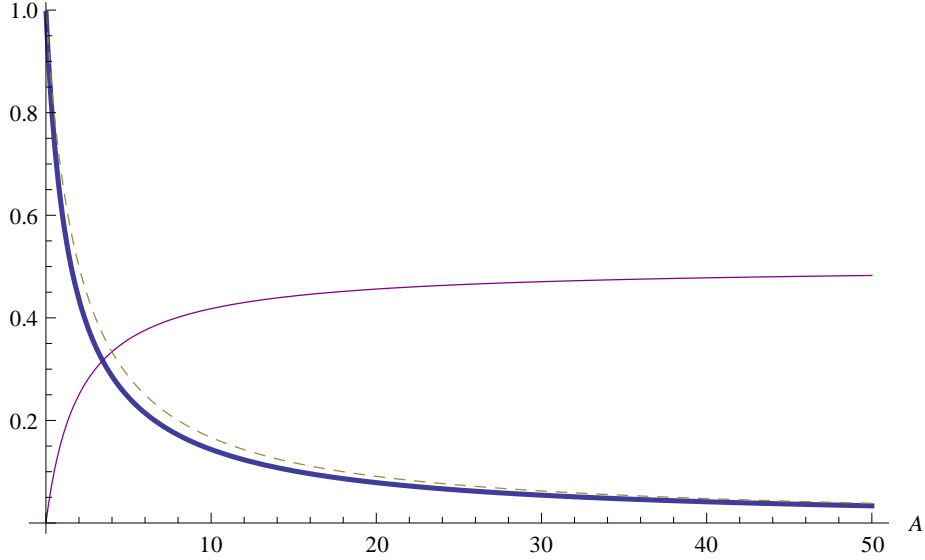


Figure 5: The same qualitative agreement as in Fig. 4, seen for the expectation value  $c(A)$  (thick) and the covariance  $C_{cp}(A)$  (thin). (Also shown is the classical  $c_{\text{classical}}(A)$  for comparison.) These graphs agree with the analytical solution  $C_{cp}(A) \propto \text{const} - c(A)$ .

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